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DISCRETE TIME DYNAMIC ESTIMATION MODEL
FOR PASSENGER ORIGIN/DESTINATION MATRICES
ON TRANSIT NETWORKS

by

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CENTRE DE RECHERCHE SUR LES TRANSPORTS



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ABSTRACT

This paper addresses the problem of estimating or updating a passenger trip matrix for transit networks from passenger counts. An extension of a model, recently proposed by Nguyen and Pallottino (1986), to take into account time information contained in the passenger counts is developed. Several promising optimization formulations of the resulting model are presented, and implementation issues examined.

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RÉSUMÉ

Nous étudions dans cet article le problème de l'estimation et de la mise à jour d'une matrice de déplacement des passagers sur un réseau de transport public à partir des comptages de trafic. Nous proposons une extension dynamique d'un modèle développé récemment par Nguyen et Pallottino (1986) qui exploite l'information contenue dans les temps de comptage. Nous examinons plusieurs formulations du modèle dynamique ainsi que certains problèmes reliés à son implantation.

2 Dynamic Trip Estimation Model

1. INTRODUCTION

A model for estimating or updating passenger trip matrices was recently proposed by Nguyen and Pallottino (1986). An application of this model to the network of the city of Torino, Italy (Inaudi and Morello, 1986), yielded reasonably good results, but also points out the difficulty in handling the model's passenger counts *equality constraints*—the constraint set is almost always infeasible. The incompatibility of the constraints is mainly caused by measurement errors and temporal variation inherent in traffic counts, but also by the steady state assumption implicit in the assignment model adopted. To overcome this shortcoming, we investigate a dynamic variant of the estimation model based on a partition of the initial time period into several time slices. The dynamic model is fully supported by the data sets used in the above application.

First, we describe the general problem of estimating or updating an origin/destination (O/D) passenger trip matrix on transit networks, using a transit assignment framework developed by Nguyen and Pallottino (1985). In section 3, we introduce a partition of the initial time period for the passenger trip matrix into several time slices, and develop the fundamental relationship between a link count for a given time slice and trips associated with different time slices. In section 4, we describe an optimization formulation based on the traditional *entropy-maximizing* approach and constraint relaxation. Alternative formulations based on statistical inference techniques, which do not impose deterministic passenger count constraints are briefly reviewed. Finally, salient implementation issues of the discrete time dynamic estimation model are also examined.

2. ESTIMATING O/D FLOWS ON TRANSIT NETWORKS

Consider an urban public transportation system consisting of a set of distinct transit lines, and stops where passengers board and alight transit carriers (buses). This transit system is abstracted into a graph, in which each transit line is identified by a unique sequence of arcs. Each stop, either served by a single transit line or by multiple lines, is represented by a subgraph, in which each line is connected to an artificial node by a pair of boarding and alighting arcs. This artificial node is referred to in the sequel as a *stop node*. Finally, walking arcs connect centroids to stop nodes. Pedestrian paths are integrated into the transit system as transit lines with appropriate characteristics.

Let $(\mathcal{N}, \mathcal{L})$ denote the modelling graph, in which $\mathcal{N}$ is the set of nodes (centroids, stops, line nodes) and $\mathcal{L}$ is the set of arcs (line segments, boarding and alighting arcs, centroid connectors). In this graph, the subset of stop nodes $\mathcal{N}^B \subset \mathcal{N}$ and that of boarding arcs $\mathcal{L}^B \subset \mathcal{L}$ assume a predominant role.

It is assumed that each transit line has a fixed line frequency, and this frequency will be associated with all the boarding arcs to this transit line. Also, an average in-vehicle travel time t_{ij} and a generalized cost c_{ij} are associated with every link $(i, j) \in \mathcal{L}$. We will restrict the analysis to the case with constant average time and cost, since the congestion effect in transit networks does not seem to be well understood at the present time.

A node where trips originate and/or terminate is called centroid. Let f_{rs} denote the average number of trips going from centroid r to centroid s , within a given time period—arriving at destination before the end of period, independent of the departure time. Let F be an array with components f_{rs} . F is usually referred to as an origin-destination trip matrix.

Consider a passenger waiting at a stop i served by several transit lines which can transport this passenger to his destination s . The decision problem faced by the passenger is whether to board an arriving carrier or to wait for a faster one. From the assignment point of view, the problem reduces to that of determining the subset of transit lines at stop i , such that the passenger always boards the first arriving carrier of this set, to get to destination s . This subset of transit lines is often referred to as the passenger's attractive set. The above adaptive component of choice behavior (i.e., to board the first arriving carrier of a subset of perceived equivalent lines) cannot be captured by the standard car network assignment framework, in which an alternative is represented by a single path connecting a pair of centroids. Clearly, a redefinition of the concept of alternative for a trip maker on a transit network must be made. In the following, an alternative is defined as an acyclic subgraph with given arc traversal probabilities—a hyperpath. More precisely, consider a single pair of centroids, say pair (r, s) .

DEFINITION. A subgraph $H = (X, E, \Pi)$, where $X \subset \mathcal{N}$, $E \subset \mathcal{L}$, and $\Pi = (\pi_{ij})$ a real value vector of dimension $|\mathcal{L}|$, is a hyperpath connecting r and s if:

- H is acyclic with at least one arc,
- node r has no predecessor and s no successor,
- for every node $i \in X - \{r, s\}$, there is a path from r to s traversing i ,

4 Dynamic Trip Estimation Model

■ the characteristic vector Π satisfies:

$$\begin{aligned}\sum_j \pi_{ij} &= 1, & \forall i \in X \\ \pi_{ij} &\geq 0, & \forall (i,j) \in \mathcal{L} \\ \pi_{ij} &= 0, & \forall (i,j) \notin E.\end{aligned}$$

For every stop node i of hyperpath H , the set of boarding arcs $B_i \subset E$ identifies the attractive set of transit lines for the subgroup of passengers who travel on this hyperpath. Each component π_{ij} represents the probability that arc (i,j) is traversed by a passenger who arrived at node i . In particular, under the following simplifying assumptions:

ASSUMPTIONS.

1. Passengers arrive randomly at every stop node,
2. All transit lines are statistically independent with a given exponential distribution of headways with mean equal to the inverse of line frequency,

the conditional probabilities π_{ij} take on specific values for boarding arcs E^B :

$$\pi_{ij} = \begin{cases} \varphi_{ij}/\Phi_i, & \forall (i,j) \in E^B \\ 1, & \forall (i,j) \in E - E^B \\ 0, & \forall (i,j) \notin E \end{cases}$$

where φ_{ij} is the frequency of the transit line associated with boarding arc (i,j) , and

$$\Phi_i = \sum_{(i,j) \in B_i} \varphi_{ij}$$

is the combined frequency of the attractive set at stop node i , identified by the set of boarding arcs B_i .

Note that in the present case, the mean wait time ρ_i , at stop node i , is equal to the inverse of the combined frequency: $\rho_i = \Phi_i^{-1}$; and the flow of passengers on hyperpath H at the stop node i is subdivided among the lines of this attractive set in proportion to their frequency. Similar assumptions and results can also be found in Chriqui and Robillard (1975) and Spiess (1983b).

Let d_l denote the probability of travelling on any particular elementary path l of hyperpath H , then:

$$d_l = \prod_{(i,j) \in E} \pi_{ij}^{\delta_{ijl}}, \quad \forall l$$

in which δ_{ijl} equals 1 if path l traverses arc (i, j) , and 0 otherwise.

The path probabilities induce, in turn, the probabilities of traversing node i :

$$b_i = \sum_l \delta_{il} d_l, \quad \forall i$$

and that of traversing arc (i, j) :

$$a_{ij} = \sum_l \delta_{ijl} d_l, \quad \forall (i, j). \quad (1.1)$$

The node and arc probabilities satisfy the following equations (Nguyen and Pallottino, 1985):

$$\begin{aligned} b_r &= b_s = 1, \\ b_j &= \sum_i b_i \pi_{ij}, \quad \forall j \in X - \{r\} \\ a_{ij} &= b_i \pi_{ij}, \quad \forall (i, j) \in E. \end{aligned} \quad (1.2)$$

These equations show clearly that the arc traversal probabilities a_{ij} can be calculated in a topological order, which obviates the explicit enumeration of elementary paths in hyperpath H . Without this property, the hyperpath framework would not be operational.

Let U_k denote the average travel cost of hyperpath H_k . This satisfies:

$$U_k = \sum_{(i,j)} a_{kij} w_{kij}, \quad (1.3)$$

where a_{kij} and w_{kij} are, the arc traversal probability and the travel cost of arc (i, j) of hyperpath H_k respectively. w_{kij} may be defined as:

$$w_{kij} = \begin{cases} \alpha \rho_i, & \forall (i, j) \in E_k^P \\ c_{ij}, & \forall (i, j) \in E_k - E_k^P \\ 0, & \forall (i, j) \notin E_k. \end{cases}$$

6 Dynamic Trip Estimation Model

where α is a user defined conversion parameter.

Now, let p_{krs} denote the proportion of passenger demand f_{rs} traveling on hyperpath H_k . The resulting link flows are:

$$v_{ij}^{rs} = \sum_k a_{kij} p_{krs} f_{rs},$$

yielding, for all O/D pairs, the following aggregated link flows:

$$v_{ij} = \sum_{rs} \sum_k a_{kij} p_{krs} f_{rs}. \quad (1.4)$$

Assume that passenger flows on a subset $M \subseteq \mathcal{L}$ have been observed. Let $\hat{V}$ denote the vector of these counts. Assume further that additional information on trip demand F —such as 'ad hoc' sample survey estimates or outdated estimates—is also available, the estimation problem addressed here can be stated broadly as:

TRIP DEMAND ESTIMATION PROBLEM. Determine an estimate of the O/D trip demand matrix F by efficiently combining passenger count based data and all other available information.

Clearly, any mathematical formulation for this problem must use the fundamental relationship (1.4). The trip proportions p_{krs} in these equations depend on the assignment model adopted. For instance, in the all or nothing assignment model, assuming that the minimum cost hyperpath—with respect to generalized costs—between any given pair of centroids is uniquely defined, the trip proportions p_{krs} satisfy:

$$p_{krs} = \begin{cases} 1, & \text{if } k \text{ is the minimum cost hyperpath connecting } r \text{ and } s \\ 0, & \text{otherwise.} \end{cases} \quad (1.5)$$

On the other hand, if the perceived cost U_k incurred on hyperpath k is assumed to be a random variable—reflecting the variation and error in the trip maker's perception of travel time or cost:

$$\tilde{U}_k = U_k + \varepsilon_k,$$

where U_k is the average cost (1.3) and ε_k is the unobservable residual, then the average trip proportion p_{krs} is given by the probability of choosing hyperpath k among alternative hyperpaths connecting r and s :

$$p_{krs} = \text{Prob}(\tilde{U}_k < \tilde{U}_h, \forall h \neq k). \quad (1.6)$$

The value of p_{krs} depends solely on the probability distribution assumed for the error terms ε_k , and may be computed independently from the trip demand F . For instance, if the error terms ε_k are assumed to be independent Weibull variates: $\varepsilon_k \sim W(0, \sigma)$, the equation (1.6) reduces to the well known *multinomial logit* model (see, for example, Dial, 1971; Domencich and McFadden, 1975):

$$p_{krs} = \frac{e^{-\theta U_k}}{\sum_h e^{-\theta U_h}},$$

where the parameter θ is defined as:

$$\theta = \frac{\pi}{\sigma\sqrt{6}}.$$

Alternatively, if the errors ε_k are assumed to follow a joint multivariate normal distribution with moments:

$$\begin{aligned} E(\varepsilon_k) &= 0, \\ \text{Var}(\varepsilon_k) &= \beta U_k, \\ \text{Cov}(\varepsilon_k, \varepsilon_h) &= \beta U_{k \cap h}, \end{aligned}$$

where $U_{k \cap h}$ is the travel cost of the common part of hyperpaths k and h , the mean proportions are then given by the *probit* model (see Daganzo and Sheffi, 1977; Daganzo, 1979).

For the sake of clarity, we will assume in the sequel that only a *single* (aggregated) hyperpath is utilized between each O/D pair with link traversal probabilities:

$$\lambda_{irs} = \sum_k a_{kl} p_{krs}, \quad \forall l = (i, j) \in M \quad (1.7)$$

leaving the choice of an assignment model to the user.

3. PARTITIONING THE TIME COORDINATE INTO INTERVALS

In this section, we develop the fundamental relationship between a link flow of a given time interval and trips associated with different time intervals. This requires a careful choice of the notation used.

Divide the time coordinate into intervals of length μ indexed by τ , for $\tau = 1, \dots, \tau_{max}$. 20 minutes would be a typical value for the interval length μ . Let $t^-(\tau)$ and $t^+(\tau)$ denote the time origin and end of interval τ respectively. The origin of the first interval, and the end of the last one will be identified later. Naturally, the choice of μ must be supported by the data available. Assume that for every link $l = (i, j) \in M$ there is a set of passenger counts $\hat{v}_l^\tau$, for $\tau = 1, \dots, \tau_{max}$ (each component $\hat{v}_l^\tau$ is equal to the number of passengers observed on link l during interval τ).

Assume further that the initial time period $[a_0, a_1]$ is a multiple of μ :

$$a_1 - a_0 = n\mu,$$

where n is a positive integer, thus $t^+(\tau_{max})$ may be set equal to a_1 . Let S_{ls} denote the effective travel time from origin node of link l to destination s ; for the all-or-nothing assignment model, S_{ls} is the travel time from l to s measured on the minimum cost hyperpath, while for all other models, S_{ls} is the travel time measured on the utilized aggregated hyperpath. Let $I(S_{ls})$ denote the number of intervals covered by S_{ls} :

$$I(S_{ls}) = \left\lfloor \frac{S_{ls}}{\mu} \right\rfloor, \quad (2.1)$$

thus, if a passenger is observed on link l in interval τ , he will reach destination s in interval $(\tau + I(S_{ls}))$. We now decompose the initial trip matrix F into a sum of n matrices F^k :

$$F = \sum_{k=1}^n F^k,$$

where f_{rs}^k is equal to the number of passengers from O/D pair (r, s) , who reach destination s in time interval

$$\tau_{max} - n + k.$$

Define

$$S_{max} = \max_{l \in M} \max_s S_{ls}, \quad (2.2)$$

then any passenger observed on link l before time $(a_0 - S_{max})$ will reach his destination prior to a_0 , thus, we may set the origin of the time coordinate at

$$t^-(1) = a_0 - \mu I(S_{max}),$$

and therefore

$$\tau_{max} = n + I(S_{max}).$$

Note that a passenger who is observed on link l in interval τ , for any $1 \leq \tau \leq \tau_{max}$, may still reach his destination prior to or after interval $[a_0, a_1]$. These passengers will be associated with residual matrix F^0 . We now have all the necessary ingredients to state the relationship between link flow v_l^r and various trip demand f_{rs}^k :

$$v_l^r = \sum_{rs} \lambda_{lrs} f_{rs}^{m_s(\tau, l)}, \quad \text{for all } 0 \leq \tau \leq \tau_{max}, \text{ and } l \in M \quad (2.3)$$

where

$$m_s(\tau, l) = \begin{cases} \tau + I(S_{ls}) - I(S_{max}), & \text{if } 1 \leq \tau + I(S_{ls}) - I(S_{max}) \leq n \\ 0, & \text{otherwise.} \end{cases}$$

4. MODEL FORMULATIONS

In this section, we will examine both the standard maximum entropy formulation, and optimization formulations based on the maximum likelihood approach.

Formulations with deterministic constraints. The standard optimization formulation with deterministic constraints can be stated as follows (Nguyen, 1983):

$$\text{Min } C(F, Y) \quad (3.1)$$

$$\text{subject to } \sum_{rs} \lambda_{lrs} f_{rs}^{m_s(\tau, l)} = \hat{v}_l^r, \quad \text{for all } 0 \leq \tau \leq \tau_{max}, \text{ and } l \in M \quad (3.2)$$

$$\sum_{k=0}^n \sum_{rs} f_{rs}^k = \text{constant},$$

$$F \geq 0,$$

$$F \in \Gamma, \quad (3.3)$$

10 Dynamic Trip Estimation Model

where $Y = (y_{rs}^k)$ represents the a priori information on trip demand F , and constraint (3.3) will take into account any additional information on F which can be expressed by linear constraints. The analytical objective function $C(F, Y)$ will depend on the approach followed. For instance, the entropy-maximizing or minimum information method (Wilson, 1970; Van Zuylen and Willumsen, 1980) yields

$$C(F, Y) = \sum_{k=0}^n \sum_{rs} f_{rs}^k \ln \frac{f_{rs}^k}{y_{rs}^k}. \quad (3.4)$$

On the other hand, in the constrained maximum likelihood approach, if Y represents a set of sample counts with given sampling fraction θ , then under the assumption that the likelihood $\mathcal{L}(Y | F)$ of observing Y given F can be expressed as a product of Poisson probability functions with parameters θf_{rs}^k , we obtain the following expression for the log-likelihood

$$\ln \mathcal{L}(Y | F) = \sum_{k=0}^n \sum_{rs} [y_{rs}^k \ln(\theta f_{rs}^k) - \theta f_{rs}^k] + \text{constant}, \quad (3.5)$$

yielding the objective function

$$C(F, Y) = - \sum_{k=0}^n \sum_{rs} [y_{rs}^k \ln(\theta f_{rs}^k) - \theta f_{rs}^k]. \quad (3.6)$$

The above constrained maximum likelihood approach was previously proposed by Landau *et al.* (1982) and by Spiess (1983a). Note that with the above objective function (3.6), a zero cell y_{rs}^k does not force the corresponding f_{rs}^k to zero. This is an advantage over the previous maximum entropy approach. On the other hand, Ben-Akiva *et al.* (1985) found that a maximum entropy trip estimation model for a single transit line, with a Bayesian seeding of nonstructural zero cells of Y (see, for example, Kirby and Leese, 1978), is more effective than constrained maximum likelihood and least-squares models.

Frequently in practice, due to measurement errors and temporal variations, the constraint set (3.3) does not have any feasible solution. This requires the use of a cumbersome data smoothing procedure. We prefer a relaxation approach, which replaces the equality

constraints (3.3) by two sets of inequalities:

$$\begin{aligned} \sum_{rs} \lambda_{lrs} f_{rs}^{m*}(\tau, l) &\leq \hat{v}_l^r + \xi_l, \\ \sum_{rs} \lambda_{lrs} f_{rs}^{m*}(\tau, l) &\geq \hat{v}_l^r - \xi_l, \end{aligned} \quad (3.7)$$

where ξ_l is a user defined tolerance, which can be tailored appropriately to reflect variation in the accuracy of passenger counts. The main disadvantage of this relaxation scheme is an increase in the number of constraints, although this should not produce significant additional computational effort.

Formulations with extended objective function. To take into account measurement errors, random variation in trip demand and the trip maker route selections over time, and inherent errors in the network and assignment models, we may assume that the passenger counts $\hat{v}_l^r$, $l \in M$, are observations of random variables with mean values equal to values predicted by the assignment predicted values v_l^r . This can be expressed as:

$$\hat{v}_l^r = \sum_{rs} \lambda_{lrs} f_{rs}^{m*}(\tau, l) + \varepsilon_l, \quad E(\varepsilon_l) = 0. \quad (3.8)$$

Assume further that the two sets of data, Y and $\hat{V}$, are statistically independent, so that the likelihood of observing both sets can be expressed as:

$$\mathcal{L}(Y, \hat{V} \mid F) = \mathcal{L}(Y \mid F) \cdot \mathcal{L}(\hat{V} \mid F). \quad (3.9)$$

The maximum likelihood estimator F^{ML} of trip demand F may then be obtained by maximizing expression (3.9) or, more conveniently, its natural logarithm:

$$F^{ML} = \arg \max_{F \in \Gamma} \left\{ \ln \mathcal{L}(Y \mid F) + \ln \mathcal{L}(\hat{V} \mid F) \right\}, \quad (3.10)$$

where Γ is the set of feasible F . Unless explicitly specified, Γ will simply be the set of all nonnegative F .

The computation of F^{ML} will depend on the probability distribution assumed for the sample count vector Y and that of the passenger count vector $\hat{V}$ (or equivalently the

12 Dynamic Trip Estimation Model

residual ϵ in equation (3.8)). In practice, the Poisson is, perhaps, the prevailing distribution choice for the error ϵ (Van Zuylen and Branston, 1982), yielding

$$\ln \mathcal{L}(\hat{V} | F) = \sum_{r=1}^{r_{max}} \sum_{i \in M} (\hat{v}_i^r \ln v_i^r - v_i^r), \quad (3.11)$$

where v_i^r is given by expression (2.3). In addition to Poisson distribution, multivariate normal distribution has also been considered, giving rise to a quadratic expression for the log-likelihood of $\hat{V}$ (see, for example, Cascetta, 1984). If the first term in the objective function of problem (3.10) is given by (3.5), then, in either cases, we have a convex program subject to simple constraints (non-negativity of F). For a detailed derivation of alternative objective functions, see the review by Nguyen and Cascetta (1986).

Aggregation level. The dynamic model was formulated with $n + 1$ trip matrices. For real life networks and typical value of n (say 3 or 4), we will be confronted with a very large number of variable f_{rs}^k . This leads us to consider aggregating the partial matrices F^k , $k = 1, \dots, n$, into fewer matrices.

In general, an aggregation of trip matrices may necessitate a corresponding aggregation of constraints in order to preserve the feasibility of the constraint set. More precisely, in terms of the constraint matrix A , if, say columns k_1 and k_2 are fused together, then all rows i with components λ_{ik_1} and λ_{ik_2} should also be fused together.

Algorithms. Problem (3.1) with objective function (3.4) or (3.5) are convex programs with linear constraints, and therefore can be solved using adaptations of a large variety of existing mathematical programming techniques (see, for example, Luenberger, 1973). However, with the entropy function (3.4), we can use Bregman's very efficient algorithm (Bregman, 1967; see also Lamond and Stewart, 1981), which is particularly advantageous for problems with large numbers of equality and inequality constraints.

With stochastic passenger counts data, and under Poisson assumptions, we have a convex program with simple constraint (3.10). This problem may be solved efficiently with an adaptation of Bertsekas's projected Newton algorithm (Bertsekas, 1982). Numerical experimentation with these methods are presently being carried out.

5. CONCLUSION

5. CONCLUSION

This paper presents a dynamic model for estimating or updating an O/D trip matrix for transit networks. Several optimisation formulations based on the entropy-maximizing and the maximum likelihood approaches are described. Potential efficient solution algorithms for the maximum entropy model with inequality constraints, and the simple constrained maximum likelihood model with Poisson distributions are pointed out. From a practical point of view, the variant with only two trip matrices (F^1 and the residual F^0), may prove to be the most effective model, and produce significantly more accurate estimates than those obtained from previously proposed models.

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